3.1

Planting the Seeds
Exploring Cubic Functions

LEARNING GOALS

In this lesson, you will:

- Represent cubic functions using words, tables, equations, and graphs.
- Interpret the key characteristics of the graphs of cubic functions.
- Analyze cubic functions in terms of their mathematical context and problem context.
- Connect the characteristics and behaviors of cubic functions to its factors.
- Compare cubic functions with linear and quadratic functions.
- Build cubic functions from linear and quadratic functions.

KEY TERMS

- relative maximum
- relative minimum
- cubic function
- multiplicity

If you have ever been to a 3D movie, you know that it can be quite an interesting experience. Special film technology and wearing funny-looking glasses allow moviegoers to see a third dimension on the screen—*depth*. Three dimensional filmmaking dates as far back as the 1920s. As long as there have been movies, it seems that people have looked for ways to transform the visual experience into three dimensions.

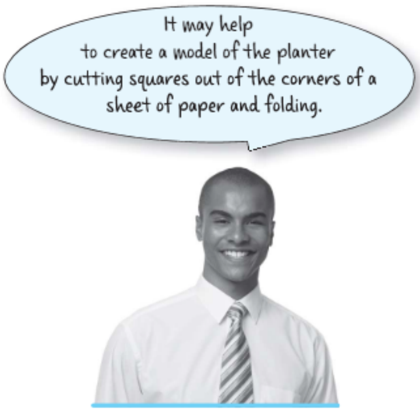
However, your brain doesn't really need special technology or silly glasses to experience depth. Think about television, paintings, and photography—artists have been making two-dimensional works of art appear as three-dimensional for a long time. Several techniques help the brain perceive depth. An object that is closer is drawn larger than a similarly sized object off in the distance. Similarly, an object in the foreground may be clear and crisp while objects in the background may appear blurry. These techniques subconsciously allow your brain to process depth in two dimensions.

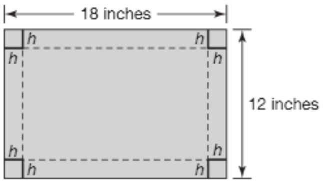
Can you think of other techniques artists use to give the illusion of depth?

PROBLEM 1 Business Is Growing

The Plant-A-Seed Planter Company produces planter boxes. To make the boxes, a square is cut from each corner of a rectangular copper sheet. The sides are bent to form a rectangular prism without a top. Cutting different sized squares from the corners results in different sized planter boxes. Plant-A-Seed takes sales orders from customers who request a sized planter box.

Each rectangular copper sheet is 12 inches by 18 inches. In the diagram, the solid lines indicate where the square corners are cut and the dotted lines represent where the sides are bent for each planter box.

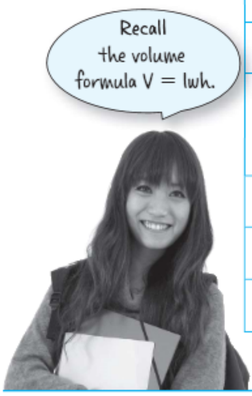




1. Organize the information about each sized planter box made from a 12 inch by 18 inch copper sheet.

   a. Complete the table. Include an expression for each planter box's height, width, length, and volume for a square corner side of length *h*.

Square Corner Side Length (inches)	Height (inches)	Width (inches)	Length (inches)	Volume (cubic inches)
0				
1				
2				
3				
4				
5				
6				
7				
h				



b. What patterns do you notice in the table?

2. Analyze the relationship between the height, length, and width of each planter box.

a. What is the largest sized square corner that can be cut to make a planter box? Explain your reasoning.

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b. What is the relationship between the size of the corner square and the length and width of each planter box?

c. Write a function $V(h)$ to represent the volume of the planter box in terms of the corner side of length h.

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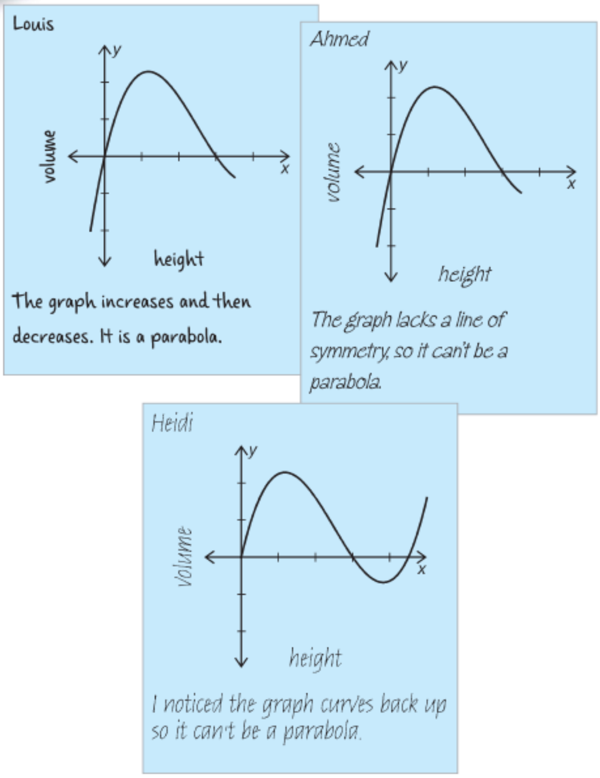
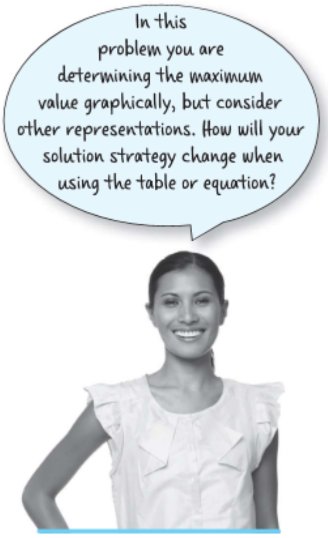
3. Louis, Ahmed, and Heidi each used a graphing calculator to analyze the volume function, $V(h)$, and sketched their viewing window. They disagree about the shape of the graph.



Evaluate each student's sketch and rationale to determine who is correct.
For the student(s) who is/are not correct, explain why the rationale is not correct.

4. Represent the function on a graphing calculator using the window $[-10, 15] \times [-400, 400]$.

   a. Describe the key characteristics of the graph?



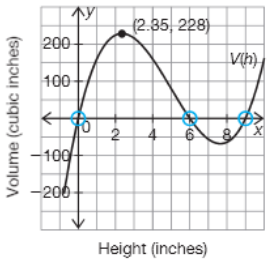
   b. What is the maximum volume of a planter box?
   State the dimensions of this planter box.
   Explain your reasoning.

   c. Identify the domain of the function *V*(*h*).
   Is the domain the same or different in terms of the context of this problem?
   Explain your reasoning.

   d. Identify the range of the function *V*(*h*).
   Is the range the same or different in terms of the context of this problem?
   Explain your reasoning.

   e. What do the *x*-intercepts represent in this problem situation? Do these values make sense in terms of this problem situation? Explain your reasoning.

The key characteristics of a function may be different within a given domain. The function $V(h) = h(12 - 2h)(18 - 2h)$ has x-intercepts at $x = 0$, $x = 6$, and $x = 9$.



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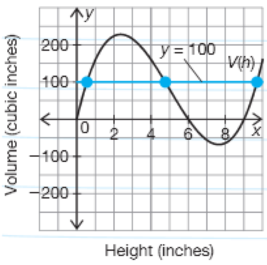
As the input values for height increase, the output values for volume approach infinity. Therefore, the function doesn't have a maximum; however, the point (2.35, 228) is a *relative maximum* within the domain interval of (0, 6). A **relative maximum** is the highest point in a particular section of a graph. Similarly, as the values for height decrease, the output values approach negative infinity. Therefore, a *relative minimum* occurs at (7.65, −68.56). A **relative minimum** is the lowest point in a particular section of a graph.

The function $v(h)$ represents all of the possible volumes for a given height h. A horizontal line is a powerful tool for working backwards to determine the possible values for height when the volume is known.

The given volume of a planter box is 100 cubic inches. You can determine the possible heights from the graph of $V(h)$.



- Draw a horizontal line at $y = 100$.
- Identify each point where $V(h)$ intersects with $y = 100$, or where $V(h) = 100$.

The first point of intersection is represented using function notation as $V(0.54) = 100$.

5. A customer ordered a particular planter box with a volume of 100 cubic inches, but did not specify the height of the planter box.

a. Use a graphing calculator to determine when $V(h) = 100$. Then write the intersection points in function notation. What do the intersection points mean in terms of this problem situation?

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b. How many different sized planter boxes can Plant-A-Seed make to fill this order? Explain your reasoning.

6. A neighborhood beautifying committee would like to purchase a variety of planter boxes with volumes of 175 cubic inches to add to business window sill store fronts. Determine the planter box dimensions that the Plant-A-Seed Company can create for the committee. Show all work and explain your reasoning.

7. Plant-A-Seed's intern claims that he can no longer complete the order because he spilled a cup of coffee on the sales ticket. Help Jack complete the order by determining the missing dimensions from the information that is still visible. Explain how you determined possible unknown dimensions of each planter box.

Plant-A-Seed
Sales Ticket

Base Area: 12 square inches
Height:
Length:
Width:
Volume:

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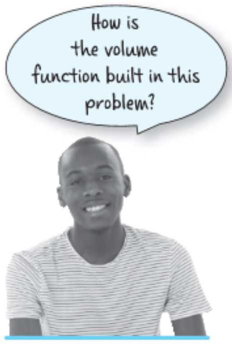
8. A customer sent the following email:

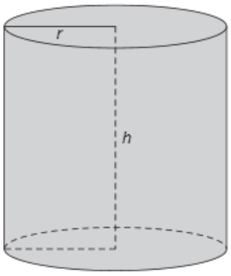
To Whom It May Concern,
I would like to purchase several planter boxes, all with a height of 5 inches. Can you make one that holds 100 cubic inches of dirt? Please contact me at your earliest convenience.
Thank you,
Muriel Jenkins



Write a response to this customer, showing all calculations.

PROBLEM 2 A Dirty Business

The Plant-A-Seed Company also makes cylindrical shaped planters for city sidewalks and store fronts. The cylindrical shaped planters come in a variety of sizes, but all have a height to radius ratio of 2:1.

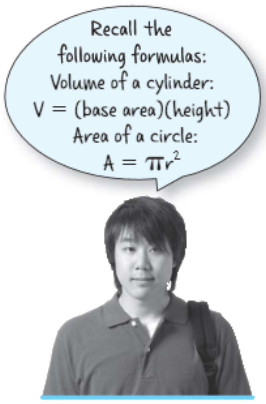


1. Why do you think Plant-A-Seed might want to manufacture different sizes of a product, but maintain a constant ratio, such as height to radius ratio of 2:1?

2. Consider different sized cylindrical planters.

   a. Complete the table.

Radius	Height (inches)	Base Area (square inches)	Volume (cubic inches)
0			
1			
2			
3			
4			
			2000
x			



   b. Describe how you determined the volume when you are given the radius.

c. Describe your method to determine the base area and the height when you are given the volume.

d. Analyze your table of values. For every unit increase in the radius, describe the rate of change in the height, area, and volume of each planter.

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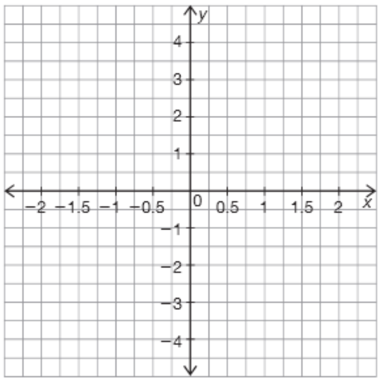
3. The base area function $A(x) = 3.14x^2$ and the height function $h(x) = 2x$ are multiplied to build the volume function $V(x) = (3.14x^2)(2x)$. Let's analyze this problem situation graphically.

a. Sketch and label the functions $h(x)$, $A(x)$, and $V(x)$ on the same coordinate plane.



b. In what ways is the graph of $V(x)$ similar to the graph of $h(x)$? In what ways is it different?

c. In what ways is the graph of $V(x)$ similar to the graph of $A(x)$? In what ways is it different?

d. Does $V(x)$ have a relative maximum or relative minimum? Explain your reasoning in terms of the function and in terms of this problem situation.

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e. Gene and Douglas disagree about the key characteristics of the graph of the cylindrical shaped planter compared to the graph of the rectangular planter box.

Gene

The volume function from the rectangular planter boxes had three x-intercepts, so the graph of the cylindrical shaped planter also must have three. If I extend my viewing window on my graphing calculator I can determine where this graph crosses the x-axis again.

Douglas

Both the linear and quadratic functions that built the volume function for the cylindrical shaped planter only cross the x-axis at (0, 0). A function can't have an x-intercept different from its factors, so (0, 0) is the only one.

Who is correct? Explain your reasoning.

The volume functions for the rectangular planter box and the cylindrical shaped planter are examples of *cubic functions*. A **cubic function** is a function that can be written in the standard form $f(x) = ax^3 + bx^2 + cx + d$ where $a \neq 0$. In other words, a cubic function is a polynomial function of degree 3.

The volume of the rectangular planter box was represented as $V(h) = h(12 - 2h)(18 - 2h)$. You can multiply the three factors to express the function in standard cubic form.

$$V(h) = h(12 - 2h)(18 - 2h)$$
$$= h(216 - 60h + 4h^2)$$
$$= 216h - 60h^3 + 4h^3$$
$$V(h) = 4h^3 - 60h^2 + 216h$$

The volume of the cylindrical shaped planter was represented as $V(x) = (3.14x^2)(2x)$. You can multiply the two factors to express the function in standard cubic form.

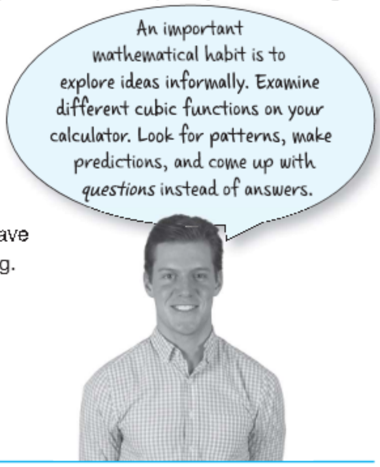
$$V(x) = 6.28x^3$$

The Fundamental Theorem of Algebra tells you that a cubic function must have 3 zeros. Roots may be any number in the set of complex numbers, and can even appear multiple times. **Multiplicity** is how many times a particular number is a zero for a given polynomial function. For example in the polynomial function that represents the volume of the cylindrical shaped planter, $V(x) = (3.14x^2)(2x)$, the zero, $x = 0$, has multiplicity 3.

4. The Fundamental Theorem states that a cubic function must have 3 zeros. Explain why the volume function in this problem crosses the x-axis only one time, yet still satisfies the Fundamental Theorem of Algebra.

5. When analyzing a table, the values in a linear function have a common first difference while quadratic functions have a common second difference. What pattern is present in cubic functions? Do you think this pattern always holds true? Explain your reasoning.

6. The graphs of linear functions are always lines while quadratic functions are always parabolas. How would you describe the shape of a cubic function? Do you think all cubic functions will have the same general shape? Explain your reasoning.



PROBLEM 3 Cubic Equivalence

Let's consider the volume formula from Problem 1, *Business is Growing*.

1. Three forms of the volume function $V(h)$ are shown.

$V(h) = h(18 - 2h)(12 - 2h)$	$V(h) = h(4h^2 - 60h + 216)$	$V(h) = 4h^3 - 60h^2 + 216h$
The product of three linear functions that represent height, length, and width.	The product of a linear function that represent the height and a quadratic function representing the area of the base.	A cubic function in standard form.

a. Algebraically verify the functions are equivalent. Show all work and explain your reasoning.

b. Graphically verify the functions are equivalent. Sketch all three functions and explain your reasoning.

c. Does the order in which you multiply factors matter? Explain your reasoning.

You can determine the product of the linear factors $(x + 2)(3x - 2)(4 + x)$ using multiplication tables.

Step 1:
Choose 2 of the binomials to multiply.
Then combine like terms.

·	**x**	**2**
$3x$	$3x^2$	$6x$
-2	$-2x$	-4

Step 2:
Multiply the product from step 1 with the remaining binomial.
Then combine like terms.

·	**4**	**x**
$3x^2$	$12x^2$	$3x^3$
$4x$	$16x$	$4x^2$
-4	-16	$-4x$

$$(x + 2)(3x - 2)(4 + x) = 3x^3 + 16x^2 + 12x - 16.$$

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2. Analyze the worked example for the multiplication of three binomials.

a. Use a graphing calculator to verify graphically that the expression in factored form is equivalent to the product written in standard form.

b. Will multiplying three linear factors always result in a cubic expression? Explain your reasoning.

3. Determine each product. Show all your work and then use a graphing calculator to verify your product is correct.

a. $(x + 2)(-3x + 2)(1 + 2x)$

b. $(10 + 2x)(5x + 7)(3x)$

4. Determine the product of the linear and quadratic factors. Then verify graphically that the expressions are equivalent.

a. $(x - 6)(2x^2 - 3x + 1)$

b. $(x)(x + 2)^2$

5. Max determined the product of three linear factors.

Max

The function $f(x) = (x + 2)^3$ is equivalent to $f(x) = x^3 + 8$

a. Explain why Max is incorrect.

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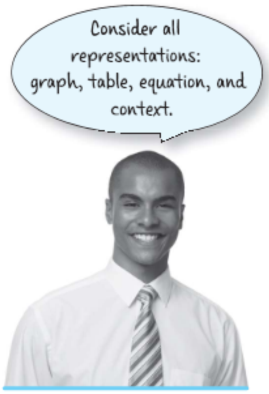
b. How many x-intercepts does the function $f(x) = (x + 2)^3$ have? How many zeros? Explain your reasoning.

Talk the Talk

In this lesson, you represented the cubic function for volume of a rectangular prism as the product of three linear functions, volume = (length)(width)(height). You also represented the cubic function for volume of a cylinder as the product of a quadratic function and a linear function, volume = (base area)(height).

1. How are cubic functions similar to linear functions? How are they different?



2. How are cubic functions similar to quadratic functions? How are they different?

Be prepared to share your solutions and methods.